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A MULTIPRESSURE REGULARIZATION FOR MULTIPHASE FLOW

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Abstract—The standard theory of ideal single-pressure multiphase fluid dynamics is known to be ill posed. To regularize this theory we dispense with the constraint of equal pressures for the different phases by introducing multiple pressures and quantities associated with interface dynamics and inertia. Via the Hamiltonian formalism, we extend the noncanonical Poisson brackets for the standard single-pressure equations to the case of multiple pressures. This formalism is used to find Lyapunov stability conditions for the regularized system. The regularized system is shown to be hyperbolic.

1. INTRODUCTION

Multiphase flow involves interpenetration of various material species. Hydrodynamic fluid models that describe such systems by using multiple velocity and density fields at a single, common pressure (recalled in section 2) are known to be ill posed and possess various types of instabilities (see e.g. Gidaspow *et al.* 1973). These difficulties are traced in Holm & Kupershmidt (1986) to the assumption of equal pressures for all the different species, in the context of the Lyapunov stability method. Here, we propose to regularize this theory by dispensing with the constraint of common pressure, which we accomplish by introducing multiple pressures, as well as additional phenomenological quantities associated with interface dynamics and inertia, in the context of the Hamiltonian formalism.

The idea of regularizing multiphase flow by introducing additional pressures is not new. However, our approach and results obtained by reasoning via the Hamiltonian formalism differ from others which introduce, e.g. viscous dissipation as in Arai (1980), numerical filtering as in Stewart (1979), surface tension as in Ramshaw & Trapp (1978), bubble inertia as in Bedford & Drumheller (1978), or phenomenological interfacial pressure jumps as in Ransom & Hicks (1984). See also Stewart & Wendroff (1984) for a recent review.

The method used here of regularizing the single-pressure, multicomponent fluid model is to extend its Poisson bracket and Hamiltonian to include multiple pressures in such a way that the known Hamiltonian structure of the multispecies fluid equations (discussed, e.g. in Iwinski & Turski (1976), Kaufman & Spencer (1982) and Holm & Kupershmidt (1983)) is recovered in the absence of interface variables. This requirement defines the Hamiltonian structure nearly uniquely. When this structure is taken together with the natural expression of the Hamiltonian as the total energy of the system, we find the desired motion equations for the extended model in section 3. In section 4 we show that for the case of two species in one space dimension our system is hyperbolic. In section 5 we use recently developed Hamiltonian methods to study the Lyapunov stability of our system for the case of two species in three space dimensions. For this case, we show that Lyapunov stable equilibrium states of the regularized equations do exist. In section 6, we determine explicitly the Lyapunov stability conditions for our extended multiphase model in the example of planar, barotropic, two-phase flow.

2 THE SINGLE-PRESSURE MODEL

The single-pressure model of ideal multiphase flow in $\mathbb{R}^n$ is described by the equations (see e.g. Stewart & Wendroff 1984)

$$\partial_t \bar{\rho}^s + \operatorname{div} \bar{\rho}^s \mathbf{v}^s = 0, \quad \partial_t v_i^s + v_j^s v_{i,j}^s = -\theta^s (\bar{\rho}^s)^{-1} P_{,i} - \Phi_{,i}, \qquad [1]$$

$$\partial_t \eta^s + \mathbf{v}^s \cdot \nabla \eta^s = 0, \qquad [2]$$

where $\bar{\rho}^s = \rho^s \theta^s$ is the macroscopic density of the sth species, ρ^s is its microscopic density, θ^s is its volume fraction, $\mathbf{v}^s$ is its velocity, η^s is its specific entropy, P is the single pressure, and Φ is the potential of an external field. Summation is assumed over all repeated subscripts, but not over the species label s. In terms of the macroscopic momentum density $\mathbf{M}^s = \bar{\rho}^s \mathbf{v}^s$, the motion equation [1b] can be written as

$$\partial_t \mathbf{M}^s + \left(\frac{\mathbf{M}^s M_j^s}{\bar{\rho}^s} \right)_{,j} = -\theta^s \nabla P - \bar{\rho}^s \nabla \Phi . \qquad [3]$$

The variables θ^s, $s = 1, \dots, N$, are to be considered as given functions of $\{\bar{\rho}^s, \eta^s\}$ through the N relations $\Sigma_{s=1}^N \theta^s = 1$, $P^1(\bar{\rho}^1/\theta^1, \eta^1) = \dots = P^N(\bar{\rho}^N/\theta^N, \eta^N) = P$, where $P^s = (\rho^s)^2 \, \partial e^s / \partial \rho^s$, with $e^s = e^s(\rho^s, \eta^s)$ being the specific internal energy of the sth species.

Equations [1]–[3] can be written in the Hamiltonian form $\partial_t F \in \{H_1, F\}_1$ with $F \in \{\bar{\rho}^s, \eta^s, \mathbf{M}^s\}$ and Poisson bracket $\{\ ,\ \}_1$ given in terms of these variables in Holm & Kupershmidt (1986):

$$\{J, I\}_1 = -\sum_s \int d^n x \left\{ \frac{\partial I}{\delta \bar{\rho}^s} \partial_j \bar{\rho}^s \frac{\delta J}{\delta M_j^s} + \frac{\partial I}{\delta \eta^s} \eta_{,j}^s \frac{\delta J}{\delta M_j^s} + \frac{\delta I}{\delta M_i^s} \left[\bar{\rho}^s \partial_i \frac{\partial J}{\delta \bar{\rho}^s} - \eta_{,i}^s \frac{\delta J}{\delta \eta^s} + (\partial_j M_i^s + M_j^s \partial_i) \frac{\delta J}{\delta M_j^s} \right] \right\}, \qquad [4]$$

where the Hamiltonian is the total energy H_1,

$$H_1 = \sum_s \int d^n x \left[\frac{|\mathbf{M}^s|^2}{2\bar{\rho}^s} + \bar{\rho}^s e^s + \bar{\rho}^s \Phi(\mathbf{x}) \right]. \qquad [5]$$

Although this model is Hamiltonian, there are two difficulties associated with it. First, [1] and [2] are well known to be ill posed even in the simplest case $n = 1$, $N = 2$, where the equations are not hyperbolic since they possess complex-valued characteristics as discussed in Gidaspow *et al.* (1973), Ramshaw & Trapp (1978), and Ransom & Hicks (1984). Second, for arbitrary spatial dimension n and number of species N the second variation of functional whose extremal points are equilibrium solutions of [1] and [2] is indefinite due to the presence of the single pressure. Thus, the Lyapunov stability of equilibrium flows for this model is prevented, as discussed in Holm & Kupershmidt (1986).

Both these difficulties with the single-pressure model can be overcome at once by allowing multiple pressures within the framework of the Hamiltonian formalism.

3. MULTIPRESSURE MODEL

To introduce the multipressure multiphase model, we postulate that:

(1) The basic dependent variables are $\{\bar{\rho}^s, \mathbf{M}^s, \eta^s\}$ as in the single-pressure model, plus $\{\theta^s\}$, $\mathbf{M}$, σ, where $\theta^s(\mathbf{x}, t)$ is sth volume fraction, as before, $\mathbf{M}$ is interfacial momentum density, and σ is interfacial mass density. Thus, the interface velocity $\mathbf{w}$ by which the volume fractions $\{\theta^s\}$ and mass density σ are transported is naturally given by $\mathbf{w} = \mathbf{M}/\sigma$. The relation $\Sigma_s \theta^s - 1 = 0$ is treated as a nondynamical constraint, i.e. this relation if initially satisfied must be preserved by the dynamics.

(2) Tensorially, the quantities $\{\theta^s\}$, $s=1, \cdots, N$, and σ which are transported by the interface velocity $\mathbf{w} = \mathbf{M}/\sigma$ are N scalar functions and a density, respectively, in analogy to $\{\eta^s\}$ and $\{\bar{\rho}^s\}$ which are transported by the velocity $\mathbf{v}^s = \mathbf{M}^s/\bar{\rho}^s$.

(3) The Poisson bracket $\{J,I\}_2$ of our system is the sum of two pieces: (a) the Poisson bracket $\{J,I\}_1$ in [4] for the variables $\{M_i,\bar{\rho}^s,\eta^s\}$, and (b) the analogous Poisson bracket for the variable $\{\mathbf{M},\sigma,\theta^s\}$:

$$\{J,I\}_2 = \{J,I\}_1 - \int d^n x \left\{ \frac{\delta I}{\delta \sigma} \partial_j \sigma \frac{\delta J}{\delta M_j} + \sum_s \frac{\delta I}{\delta \theta^s} \theta^s_{,j} \frac{\delta J}{\delta M_j} + \frac{\delta I}{\delta M_i} \left[\sigma \partial_i \frac{\delta J}{\delta \sigma} - \sum_s \theta^s_{,i} \frac{\delta J}{\delta \theta^s} + (\partial_j M_i + M_j \partial_i) \frac{\delta J}{\delta M_j} \right] \right\}. \quad [6]$$

(4) The Hamiltonian of our system, i.e. its total energy, is given by

$$H_2 = H_1 + \int d^n x \left[\frac{|\mathbf{M}|^2}{2\sigma} + \epsilon(\sigma) \right], \quad [7]$$

where $\epsilon(\sigma)$ is the energy density associated with changes of the interfacial mass density. (The Lie algebraic interpretation of both Poisson brackets [4] and [6] can be found in Holm & Kupershmidt (1983).)

These four postulates produce the following equations of motion, via the rule $F = \{H,F\}_2$ for $F \in \{\mathbf{M}^s, \bar{\rho}^s, \eta^s, \mathbf{M}, \sigma, \theta^s\}$:

$$\partial_t \bar{\rho}^s + \operatorname{div} \bar{\rho}^s \mathbf{v}^s = 0\,, \quad \partial_t \eta^s + \mathbf{v}^s \cdot \nabla \eta^s = 0\,, \quad [8]$$

$$\partial_t \mathbf{v}^s + (\mathbf{v}^s \cdot \nabla)\mathbf{v}^s = -\theta^s(\bar{\rho}^s)^{-1} \nabla P^s - \nabla \Phi\,, \quad [9]$$

$$\partial_t \theta^s + \mathbf{w} \cdot \nabla \theta^s = 0\,, \quad [10]$$

$$\partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} = -\frac{1}{\sigma} \sum_s P^s \nabla \theta^s - \epsilon''(\sigma) \nabla \sigma\,, \quad [11]$$

$$\partial_t \sigma + \operatorname{div} \sigma \mathbf{w} = 0\,. \quad [12]$$

Here $P^s = P^s(\bar{\rho}^s/\theta^s,\eta^s)$ is the thermodynamic pressure for the sth species: $P^s = (\bar{\rho}^s/\theta^s)^2 \, \partial e^s(\bar{\rho}^s/\theta^s,\eta^s)/\partial(\bar{\rho}^s/\theta^s)$, where e^s is the specific internal energy of the sth species, and Φ is the external potential. Note that [10] preserves the relation $\Sigma_s \theta^s - 1 = 0$.

For the two-species case, [8] and [9] stay unchanged, while [10]–[12] become, using $\theta^1 = \theta$, $\theta^2 = 1 - \theta$,

$$\partial_t \theta + \mathbf{w} \cdot \nabla \theta = 0\,, \quad [10']$$

$$\partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} = \frac{1}{\sigma}(P^2 - P^1)\nabla\theta - \nabla \epsilon'(\sigma)\,, \quad [11']$$

$$\partial_t \sigma + \operatorname{div} \sigma \mathbf{w} = 0\,. \quad [12']$$

For separated flow, [10′] describes transport of volume fraction θ by the interface with velocity $\mathbf{w}$, whose acceleration is given in [11′] in the form of Newton's law, with an inertial mass density σ, which by [12′] is also transported (as a density) by the interface. Equations [10′]–[12′] were originally proposed by F. Harlow and B. Wendroff and are mentioned in Stewart & Wendroff (1984). Recently, Wendroff (private communication) has derived a system of equations similar to [8], [9], and [10′]–[12′] in one dimension for three-layer channel flow for the limit in which the thickness of the intermediate layer tends to zero, but the product of thickness times density tends to $\sigma \neq 0$. Thus, Wendroff's treatment produces a pressure jump ($P^2 \neq P^1$) and equations similar to the present model by treating the interface as an additional, hydrodynamically inert fluid, with nonzero mass density and transport velocity, but zero volume fraction.

Notice that in the case where the first $N-1$ species, say, are dispersed in species number N, [11] written as

$$\partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} = \frac{1}{\sigma} \sum_{s=1}^{N-1} (P^N - P^s)\nabla \theta^s - \nabla \epsilon'(\sigma) \qquad [11'']$$

also has the form of the Newtonian force law for interface acceleration.

Conserved quantities for the full system [8]–[12] in three dimensions are

$$C^s = \int d^3x \, \bar{\rho}^s F^s(\eta^s, q^s) \,. \qquad [13a]$$

In [13a], the quantity q^s defined by

$$q^s = (\bar{\rho}^s)^{-1} \operatorname{curl} \mathbf{v}^s \cdot \nabla \eta^s \qquad [13b]$$

is the macroscopic potential vorticity for the sth species, which satisfies

$$\partial_t q^s + \mathbf{v}^s \cdot \nabla q^s = 0 \,, \qquad [13c]$$

and F^s are arbitrary functions.

Additional conserved quantities of the same form are

$$K^s = \int d^3x \, \sigma \, G^s(Q^s, \theta^s) \,, \qquad [14a]$$

where the quantity Q^s, defined by

$$Q^s = \sigma^{-1} \operatorname{curl} \mathbf{w} \cdot \nabla \theta^s \,, \qquad [14b]$$

satisfies

$$\partial_t Q^s + \mathbf{w} \cdot \nabla Q^s = 0 \,, \qquad [14c]$$

and G^s are arbitrary functions.

Remark. In the multispecies case with $N > 2$, the notion of interface velocity $\mathbf{w}$ may be generalized for any pair of adjacent materials, by introducing $N(N-1)/2$ interface velocities $\mathbf{w}^{\alpha\beta} = \mathbf{w}^{\beta\alpha}$, $\alpha \neq \beta$, where α and β take values 1, 2, ···, N. Correspondingly $\sigma^{\alpha\beta} = \sigma^{\beta\alpha}$ are the mass densities associated to the interfaces; and antisymmetric variables $\phi^{\alpha\beta} = -\phi^{\beta\alpha}$ are introduced such that $\theta^s - 1/N = \Sigma_\alpha \phi^{s\alpha}$. The motion equations for the new system (no sum on repeated superscripts unless explicitly stated) are

$$\partial_t \bar{\rho}^s + \operatorname{div} \bar{\rho}^s \mathbf{v}^s = 0 \,, \quad \partial_t \eta^s + \mathbf{v}^s \cdot \nabla \eta^s = 0 \,, \qquad [15]$$

$$\partial_t \mathbf{v}^s + (\mathbf{v}^s \cdot \nabla)\mathbf{v}^s = (\bar{\rho}^s)^{-1}\left(\frac{1}{N} + \sum_\alpha \phi^{s\alpha}\right)\nabla P^s - \nabla \Phi \,,$$

$$\partial_t \sigma^{\alpha\beta} + \operatorname{div} \sigma^{\alpha\beta}\mathbf{w}^{\alpha\beta} = 0 \,, \quad \partial_t \phi^{\alpha\beta} + \mathbf{w}^{\alpha\beta} \cdot \nabla \phi^{\alpha\beta} = 0 \,,$$

$$\partial_t \mathbf{w}^{\alpha\beta} + (\mathbf{w}^{\alpha\beta} \cdot \nabla)\mathbf{w}^{\alpha\beta} = (\sigma^{\alpha\beta})^{-1} (P^\beta - P^\alpha)\nabla \phi^{\alpha\beta} - \epsilon^{\alpha\beta\,\prime\prime}(\sigma^{\alpha\beta})\nabla \sigma^{\alpha\beta}$$

These equations form a Hamiltonian system with Poisson bracket

$$\{J,I\}_3 = \{J,I\}_1 - \sum_{\alpha<\beta} \int d^n x \left\{ \frac{\delta I}{\delta \sigma^{\alpha\beta}} \partial_j \sigma^{\alpha\beta} \frac{\delta J}{\delta M_j^{\alpha\beta}} + \frac{\delta I}{\delta \phi^{\alpha\beta}} \phi_{,i}^{\alpha\beta} \frac{\delta J}{\delta M_i^{\alpha\beta}} \right.$$

$$\left. + \frac{\delta I}{\delta M_i^{\alpha\beta}} \left[\sigma^{\alpha\beta} \partial_i \frac{\delta J}{\delta \sigma^{\alpha\beta}} - \phi_{,i}^{\alpha\beta} \frac{\delta J}{\delta \phi^{\alpha\beta}} + (\partial_j M_i^{\alpha\beta} + M_j^{\alpha\beta} \partial_i) \frac{\delta J}{\delta M_j^{\alpha\beta}} \right] \right\}, \quad [16]$$

and with Hamiltonian

$$H_4 = H_1 + \sum_{\alpha<\beta} \int d^n x \left[\frac{|\mathbf{M}^{\alpha\beta}|^2}{2\sigma^{\alpha\beta}} + \epsilon^{\alpha\beta}(\sigma^{\alpha\beta}) \right], \quad [17]$$

where $\mathbf{M}^{\alpha\beta} = \mathbf{w}^{\alpha\beta}\sigma^{\alpha\beta}$ (no sum on α,β) and $\epsilon^{\alpha\beta}$ is the internal energy density of the $\alpha\beta$ interface.

The two-species case ($N = 2$) is recovered from the system [15] when σ^{12} is identified with σ, ϕ^{12} with $\frac{1}{2}(\theta^1 - \theta^2)$, and $\mathbf{M}^{12}$ with $\sigma\mathbf{w}$.

4 HYPERBOLICITY OF THE MULTIPRESSURE MODEL IN ONE SPACE DIMENSION

In this section, we show that in the case $n = 1$, $N = 2$, the equations [8]–[12] provide a hyperbolic regularization of the corresponding single-pressure equations. Let

$$\mathbf{U}^T = (\bar{\rho}^1, \bar{\rho}^2, v^1, v^2, \sigma, \phi, w, \eta^1, \eta^2),$$

where $\phi = \theta^1 - \frac{1}{2}$, and let

$$\mathbf{A} = \begin{vmatrix} v^1 & 0 & \bar{\rho}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v^2 & 0 & \bar{\rho}^2 & 0 & 0 & 0 & 0 & 0 \\ (c^1)^2/\bar{\rho}^1 & 0 & v^1 & 0 & 0 & -(c^1)^2/(\phi+1/2) & 0 & 0 & 0 \\ 0 & (c^2)^2/\bar{\rho}^2 & 0 & v^2 & 0 & (c^2)^2/(\phi-1/2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w & 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon''(\sigma) & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^2 \end{vmatrix}, \quad [18]$$

where $(c^s)^2 = \partial P^s/\partial \rho^s$, $\rho^s = \bar{\rho}^s/\theta^s$, $s = 1, 2$, and $\epsilon''(\sigma) = d^2\epsilon(\sigma)/d\sigma^2$. Then [8]–[12] may be written, neglecting $\Phi(x)$, as

$$\partial_t \mathbf{U} + \mathbf{A}\partial_x \mathbf{U} = 0. \quad [19]$$

Observe that the characteristic polynomial of $\mathbf{A}$,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = [(v^1-\lambda)^2 - (c^1)^2][(v^2-\lambda)^2-(c^2)^2]$$
$$\times (w-\lambda)[(w-\lambda)^2 - \sigma\epsilon''(\sigma)](v^1-\lambda)(v^2-\lambda), \quad [20]$$

has real roots

$$\begin{aligned} \lambda_{1,2} &= v^1 \pm c^1, \\ \lambda_{3,4} &= v^2 \pm c^2, \\ \lambda_5 &= w, \\ \lambda_{6,7} &= w \pm \sqrt{\sigma\epsilon''(\sigma)}, \\ \lambda_8 &= v^1, \ \lambda_9 = v^2, \end{aligned} \quad [21]$$

provided

$$\epsilon''(\sigma) \geq 0 , \qquad [22]$$

which is a natural requirement for an internal energy. Thus, the multipressure model in one space dimension has all real characteristics with, in general, distinct eigenvalues when [22] holds strictly. Hence, the system [8]–[12] is hyperbolic.

5 LYAPUNOV STABILITY CONDITIONS FOR STEADY ADIABATIC SOLUTIONS IN THREE DIMENSIONS

Now we will show for the two-species case that extremal points of the sum

$$H_C = H_2 + \sum_{s=1}^{2} (C^s + K^s) , \qquad [23]$$

with H_2, C^s, K^s, given in [7], [13a], and [14a], respectively, are equilibrium states of the multipressure multiphase equations [8]–[12] in three dimensions. Stability of these steady states is then investigated by studying the conditions for definiteness of the second variation of H_C, denoted $\delta^2 H_C$, evaluated at the equilibrium state. Note that $\delta^2 H_C$ is preserved by the linearized equations about the equilibrium point (see e.g. Holm *et al* 1985, appendix A). Thus, when $\delta^2 H_C$ is definite in sign, it defines a conserved norm, in which the linearized equations are Lyapunov stable; see Abarbanel *et al.* (1986) and Holm *et al.* (1985) for examples of Lyapunov stability analyses using Hamiltonian structure in other cases of fluid dynamical theories.

In three dimensions for the two-species case ($n = 3$, $N = 2$), the steady states $\{\bar{\rho}^1_e, \bar{\rho}^2_e, \mathbf{v}^1_e, \mathbf{v}^2_e, \sigma_e, \theta_e, \mathbf{w}_e, \eta^1_e, \eta^2_e\}$ (where $\theta^1_e = \theta_e$, $\theta^2_e = 1-\theta_e$, since $\Sigma_s\ \theta^s_e = 1$) of the multipressure multiphase equations [8]–[12] and their implied equations [13c] and [14c] satisfy the relations

$$0 = \operatorname{div} \bar{\rho}^s_e \mathbf{v}^s_e , \qquad [24a]$$

$$0 = \mathbf{v}^s_e \cdot \nabla \eta^s_e , \qquad [24b]$$

$$0 = (\mathbf{v}^s_e \cdot \nabla)\mathbf{v}^s_e + \theta^s_e (\bar{\rho}^s_e)^{-1} \nabla P^s_e + \nabla \Phi , \qquad [24c]$$

$$0 = \mathbf{w}_e \cdot \nabla \theta_e , \qquad [24d]$$

$$0 = (\mathbf{w}_e \cdot \nabla)\mathbf{w}_e + \frac{1}{\sigma} (P^1_e - P^2_e) \nabla \theta_e + \nabla \epsilon'(\sigma_e) , \qquad [24e]$$

$$0 = \operatorname{div} \sigma_e \mathbf{w}_e , \qquad [24f]$$

$$0 = \mathbf{v}^s_e \cdot \nabla q^s_e , \qquad [24g]$$

$$0 = \mathbf{w}_e \cdot \nabla Q_e \qquad [24h]$$

By virtue of the identity $(\mathbf{v} \cdot \nabla)\mathbf{v} = -\mathbf{v} \times \operatorname{curl} \mathbf{v} + \frac{1}{2}\nabla|\mathbf{v}|^2$, [24c] and [24e] can be rewritten as

$$0 = \mathbf{v}^s_e \times \operatorname{curl} \mathbf{v}^s_e - \nabla\left(\frac{1}{2}|\mathbf{v}^s_e| + h^s_e + \Phi\right) + T^s_e \nabla \eta^s_e \qquad [24c']$$

and

$$0 = \mathbf{w}_e \times \operatorname{curl} \mathbf{w}_e - \nabla\left[\frac{1}{2}|\mathbf{w}_e|^2 + \epsilon'(\sigma_e)\right] + \frac{1}{\sigma_e} (P^2 - P^1) \nabla \theta^1_e , \qquad [24e']$$

respectively, where $h_e^s = e^s(\bar{\rho}_e^s/\theta_e^s, \eta_e^s) + P_e^s\theta_e^s/\bar{\rho}_e^s$ and $T_e^s = \partial e^s(\bar{\rho}_e^s/\theta_e^s, \eta_e^s)/\partial\eta_e^s$. In turn, [24c'] and [24e'] imply, using [24b] and [24d], that

$$\mathbf{v}_e^s \cdot \nabla\left(\frac{1}{2}|\mathbf{v}_e^s|^2 + h_e^s + \Phi\right) = 0\,, \qquad [25a]$$

$$\mathbf{w}_e \cdot \nabla\left[\frac{1}{2}|\mathbf{w}_e|^2 + \epsilon'(\sigma_e)\right] = 0\,. \qquad [25b]$$

Sufficient conditions for these two relations [25a,b] to hold are the existence of functions $K^s(\eta_e^s, q_e^s)$ and $L(\theta_e, Q_e)$ (the superscript "1" on θ_e and Q_e is dropped for the two-species case) such that

$$\frac{1}{2}|\mathbf{v}_e^s|^2 + h_e^s + \Phi = K^s(\eta_e^s, q_e^s)\,, \qquad [26a]$$

$$\frac{1}{2}|\mathbf{w}_e|^2 + \epsilon'(\sigma_e) = L(\theta_e, Q_e)\,, \qquad [26b]$$

as can be seen from [24b], [24g] and [24d], [24h], respectively. The relations [26a,b] are analogs for two-phase flow of Bernoulli's law.

Vector multiplication of [24c'] (resp. [24e']) by $\nabla\eta_e^s$ (resp. $\nabla\theta_e$) gives the following relations, provided $q_e^s \neq 0$ and $Q_e \neq 0$:

$$\bar{\rho}_e^s\mathbf{v}_e^s = \frac{1}{q_e^s}\nabla\eta_e^s \times \nabla\left(\frac{1}{2}|\mathbf{v}_e|^2 + h_e^s + \Phi\right), \qquad [27a]$$

$$\sigma_e\mathbf{w}_e = \frac{1}{Q_e}\nabla\theta_e \times \nabla\left(\frac{1}{2}|\mathbf{w}_e|^2 + \epsilon'(\sigma_e)\right). \qquad [27b]$$

Use of the relations [26a,b] then gives

$$\bar{\rho}_e^s\mathbf{v}_e^s = \frac{1}{q_e^s}K_q^s\nabla\eta_e^s \times \nabla q_e^s\,, \qquad [28a]$$

$$\sigma_e\mathbf{w}_e = \frac{1}{Q_e}L_Q\nabla\theta_e \times \nabla Q_e\,, \qquad [28b]$$

where $K_q^s := \partial K^s(\eta_e^s, q_e^s)/\partial q_e^s$ and $L_Q := \partial L(\theta_e, Q_e)/\partial Q_e$. Similarly, vector multiplying [24c'] by ∇q_e^s and [24e'] by ∇Q_e, and using relations [28] gives

$$\frac{\nabla q_e^s \cdot \operatorname{curl} \mathbf{v}_e^s}{\nabla\eta_e^s \cdot \operatorname{curl} \mathbf{v}_e^s} = \frac{T_e^s - K_\eta^s}{K_q^s}\,, \qquad [29a]$$

$$\frac{\nabla Q_e \cdot \operatorname{curl} \mathbf{w}_e}{\nabla\theta_e \cdot \operatorname{curl} \mathbf{w}_e} = \frac{\sigma_e^{-1}(P_e^2 - P_e^1) - L_\theta}{L_Q}\,. \qquad [29b]$$

These relations will be useful in demonstrating the following proposition.

Proposition. A smooth equilibrium solution $\{\bar{\rho}_e^s, \mathbf{v}_e^s, \eta_e^s, \sigma_e, \mathbf{w}_e, \theta_e^s;\ s = 1, 2$, with $q_e^s Q_e \neq 0\}$ whose velocities $\mathbf{v}_e^s$, $\mathbf{w}_e$ are tangential to the (fixed) boundary of the domain of flow is a critical point of H_C in [23], provided the following relations hold at equilibrium:

$$K^s(\eta_e^s, q_e^s) + F^s(\eta_e^s, q_e^s) - q_e^s F_q^s(\eta_e^s, q_e^s) = 0\,, \qquad [30a]$$

$$L(\theta_e, Q_e) + G(\theta_e, Q_e) - Q_e G_a(\theta_e, Q_e) = 0\,, \qquad [30b]$$

where F^s and G appear in H_C and are given in [13a] and [14a], respectively.

Remark. The relations [30a,b] connect the functions F^s and G to the Bernoulli functions K^s and the Long function L in [26a,b]. The relations [30a,b] define nondegenerate (e.g nonstatic, nonpotential) equilibrium flows

Proof. We write explicitly the conserved functional [23] as

$$H_C = \int_D d^3x\left[\frac{1}{2}\sigma|\mathbf{w}|^2 + \epsilon(\sigma) + \sigma G(\theta,Q)\right] + \sum_{s=1}^{2}\int_D d^3x\left[\frac{1}{2}\bar{\rho}^s|\mathbf{v}^s|^2 + \bar{\rho}^s e^s\left(\frac{\bar{\rho}^s}{\theta^s},\eta^s\right) + \bar{\rho}^s\Phi(\mathbf{x}) + \bar{\rho}^s F^s(\eta^s,q^s)\right] + \int_D d^3x\left(\sum\lambda^s \mathrm{curl}\,\mathbf{v}^s\cdot\nabla\eta^s + \Lambda\,\mathrm{curl}\,\mathbf{w}\cdot\nabla\theta\right), \quad [31]$$

where D is the domain of flow and λ^s, Λ are constants (separated out for convenience below in [33a,b]). After integrating by parts, we have the following expression for the first variation δH_C;

$$\begin{aligned}\delta H_C = \int_D d^3x\Bigg\{&\left[\frac{1}{2}|\mathbf{w}|^2 + \epsilon'(\sigma) - QG_Q(\theta,Q) + G(\theta,Q)\right]\delta\sigma \\ &+ \delta\mathbf{w}\cdot[\sigma\mathbf{w} + G_{QQ}\nabla Q\times\nabla\theta] + \sum_s \delta\mathbf{v}^s[\bar{\rho}^s\mathbf{v}^s + F^s_{qq}\nabla q^s\times\nabla\eta^s] \\ &+ \sum_s \delta\bar{\rho}^s\left[\frac{1}{2}|\mathbf{v}^s|^2 + h^s\left(\frac{\bar{\rho}^s}{\theta^s},\eta^s\right) + \Phi(\mathbf{x}) + F^s(\eta^s,q^s) - q^sF^s\right] \\ &+ \delta\theta[P^2 - P^1 - \mathrm{curl}\,\mathbf{w}\cdot\nabla G_Q(Q,\theta) + \sigma G_\theta(Q,\theta)] \\ &+ \sum_s\delta\eta^s[\bar{\rho}^s(e^s_\eta + F^s_\eta) - \nabla F^s_q\cdot\mathrm{curl}\,\mathbf{v}^s]\Bigg\} \\ &+ \oint_{\partial D}(G_Q + \Lambda)(-\delta\mathbf{w}\times\nabla\theta + \delta\theta\,\mathrm{curl}\,\mathbf{w})\cdot\hat{\mathbf{n}}\,dS \\ &+ \sum_s\oint_{\partial D}(F^s_q + \lambda^s)(-\delta\mathbf{v}^s\times\nabla\eta^s + \delta\eta^s\,\mathrm{curl}\,\mathbf{v}^s)\cdot\hat{\mathbf{n}}\,dS,\end{aligned} \quad [32]$$

where dS is the surface element on the boundary ∂D and $\hat{\mathbf{n}}$ is its unit normal vector. For H_C in [31] to have a critical point for steady flows, the steady-state relations [24a–h] must cause each coefficient and the boundary terms to vanish, provided F and G satisfy conditions [30a,b]. By relations [25a,b] and the tangency to the boundary of $\mathbf{v}^s_e$ and $\mathbf{w}_e$, all the equilibrium quantities θ_e, Q_e, η^s_e, q^s_e, are constants on the boundary. Hence, by choosing the constants λ^s and Λ according to

$$G_Q(\theta_e,Q_e)\big|_{\partial D} + \Lambda = 0 \quad , \quad [33a]$$

$$F^s_q(\eta^s_e,q^s_e)\big|_{\partial D} + \lambda^s = 0\,, \quad [33b]$$

the boundary terms will vanish at equilibrium.

The remaining coefficients in [32] vanish for stationary flows by virtue of the two relations [30a,b] of the proposition as follows. The $\delta\bar{\rho}^s$ and $\delta\sigma$ coefficients vanish by [30a] and [30b], respectively. Upon differentiating [30a] with respect to q_e, we get

$$\frac{K^s_q(\eta^s_e,q^s_e)}{q^s_e} = F^s_{qq}(\eta^s_e,q^s_e)\,. \quad [34]$$

Therefore, the $\delta \mathbf{v}^s$ coefficient in [32] vanishes for steady flows by [26a] and [28a]. Similarly, differentiating [30b] with respect to Q_e implies

$$\frac{L_Q(\theta_e,Q_e)}{Q_e} = G_{QQ}(\theta_e,Q_e)\,. \qquad [35]$$

So the $\delta \mathbf{w}$ coefficient in [32] vanishes for steady flows by [26b] and [28b]. Next, upon substituting [30a] into it, the $\delta\eta^s$ coefficient in [32] vanishes by relations [29a] and [34]. Similarly, the $\delta\theta$ coefficient vanishes upoon substituting [26b] into it and using [29b] and [35]. Thus, H_C in [31] has a critical point for equilibrium solutions when relations [30a,b] of the proposition hold.

The second variation of H_C at equilibrium $\delta^2 H_C$ is given by the quadratic form

$$\begin{aligned}
\delta^2 H_C = &\int d^3x \{\sigma_e|\delta\mathbf{w} + \sigma_e^{-1}\mathbf{w}_e\delta\sigma|^2 + (\epsilon''(\sigma_e) - \sigma_e^{-1}|\mathbf{w}_e|^2)(\delta\sigma)^2 \\
&+ 2G_\theta\delta\sigma\delta\theta + 2G_Q \operatorname{curl}\delta\mathbf{w}\cdot\nabla\delta\theta + 2\sigma_e G_{Q\theta}\delta\theta\delta Q \\
&+ \sigma_e G_{\theta\theta}(\delta\theta)^2 + \sigma_e G_{QQ}(\delta Q)^2\} \\
&+ \sum_s \int d^3x \Bigg\{ \bar{\rho}_e^s|\delta\mathbf{v}^s + (\bar{\rho}_e^s)^{-1}\mathbf{v}_e^s\delta\bar{\rho}^s|^2 - (\bar{\rho}_e^s)^{-1}|\mathbf{v}_e^s|\,(\delta\bar{\rho}^s)^2 \qquad [36] \\
&+ \bar{\rho}_e^s(\beta^s)^2\left[\delta\left(\frac{\bar{\rho}^s}{\theta^s}\right) + (\beta^s)^{-2}e^s_{\rho\eta}\,\delta\eta^s\right]^2 + 2(e^s_\eta + F^s_\eta)\delta\bar{\rho}^s\delta\eta^s \\
&+ 2F^s_q \operatorname{curl}\delta\mathbf{v}^s\cdot\nabla\delta\eta^s + 2\bar{\rho}_e^s F^s_{\eta q}\,\delta\eta^s\delta q^s \\
&+ \bar{\rho}_e^s\left[e^s_{\eta\eta} - \left(\frac{e^s{}_{\rho\eta}}{\beta^s}\right)^2 + F^s_{\eta\eta}\right](\delta\eta^s)^2 + \bar{\rho}_e^s F^s_{qq}\,(\delta q^s)^2 \Bigg\}\,,
\end{aligned}$$

where

$$\delta\left(\frac{\bar{\rho}^s}{\theta^s}\right) = (\theta_e^s)^{-1}\delta\bar{\rho}^s - (\theta_e^s)^{-2}\bar{\rho}_e^s\delta\theta^s\,,$$

$$\bar{\rho}_e^s\delta q^s = -q_e^s\,\delta\bar{\rho}^s + \operatorname{curl}\delta\mathbf{v}^s\cdot\nabla\eta_e^s + \operatorname{curl}\mathbf{v}_e^s\cdot\nabla\delta\eta^s\,,$$

with an analogous expression for $\sigma_e\delta Q$. Throughout, G, F^s, and their derivatives G_θ, etc. are to be evaluated at equilibrium, and $(\beta^s)^2 = (\theta_e^s c_e^s/\bar{\rho}_e^s)^2$, with $(c_e^s)^2 = \partial P_e^s/\partial(\bar{\rho}_e^s/\theta_e^s)$ being the square of the sound speed for the sth species.

A given flow characterized by the functions F^s and G will be linearly Lyapunov stable in the class of smooth solutions, provided F^s and G satisfy the conditions required for the integrand of $\delta^2 H_C$ in [36] to be positive definite. [See Holm *et al*. (1983; 1985) for detailed explanations of this technique for finding Lyapunov stability conditions.]

The derivation of the conditions for definiteness of the quadratic form in [36] is standard, but the explicit conditions are too complicated to be illuminating at this stage. In the next section, the corresponding Lyapunov stability conditions are studied for the simpler case of planar barotropic two-phase flow. We simply state at this stage that Lyapunov stable equilibrium states of the regularized equations [8]–[12] in three dimensions do exist provided (1) the gradient of the volume fraction and the gradients of the specific entropy for each species remain bounded in a manner determined by equilibrium flow quantities, and (2) certain other stability conditions are satisfied, as determined by Sylvester's criterion for definiteness of the quadratic form in [36]; see Holm *et al*. (1985) for discussions of such stability conditions in a number of fluid models. Moreover, the quadratic form in [36] reduces to its counterpart for adiabatic flow of a single fluid in three dimensions when the two fluids move together as one, i.e. when

$$\mathbf{v}^1 = \mathbf{v}^2 = \mathbf{v},\ \theta^1 = \theta^2 = \frac{1}{2},$$
$$\bar{\rho}^1 = \bar{\rho}^2 = \frac{1}{2}\rho,\quad \eta^1 = \eta^2 = \frac{1}{2}\eta,\quad \delta\theta = \delta\mathbf{w} = \delta\sigma = 0\,,$$

and $\mathbf{w}$, σ, Q are absent. In that case, the conditions for Lyapunov stability are given in Holm *et al.* (1985).

6 LYAPUNOV STABILITY ANALYSIS FOR PLANAR BAROTROPIC TWO-PHASE FLOW

In this section we consider two-dimensional barotropic two-phase flow taking place in a domain $D \subset \mathbb{R}^2$ in the x-y plane. For such motion to remain planar, each of the dependent variables $\{\bar{\rho}^s, \mathbf{v}^s, \sigma, \theta, \mathbf{w}\}$ must be a function only of (x,y,t), and $\mathbf{v}^s$, $\mathbf{w}$, and $\nabla\theta$ must lie in the x–y plane so that curl $\mathbf{v}^s$ and curl $\mathbf{w}$ will be directed normally to the plane along $\hat{\mathbf{z}}$. The equations of motion [8]–[12] for this situation become

$$\partial_t\bar{\rho}^s = -\mathrm{div}\,\bar{\rho}^s\mathbf{v}^s\,, \qquad [37a]$$

$$\partial_t\mathbf{v}^s = -(\mathbf{v}^s\cdot\nabla)\mathbf{v}^s - \nabla\left[h^s\left(\frac{\bar{\rho}^s}{\theta^s}\right) + \Phi(x,y)\right], \qquad [37b]$$

$$\partial_t\sigma = -\mathrm{div}\,\sigma\mathbf{w}\,, \qquad [37c]$$

$$\partial_t\theta = -\mathbf{w}\cdot\nabla\theta\,, \qquad [37d]$$

$$\partial_t\mathbf{w} = -(\mathbf{w}\cdot\nabla)\mathbf{w} + \frac{1}{\sigma}(P^2-P^1)\nabla\theta - \nabla\epsilon'(\sigma)\,, \qquad [37e]$$

where $\theta^1 = \theta$, $\theta^2 = 1-\theta$ for θ^s, $s = 1, 2$, and $h^s(\bar{\rho}^s/\theta^s)$ is the specific enthalpy of the sth phase, obeying $dh^s(\bar{\rho}^s/\theta^s) = (\theta^s/\bar{\rho}^s)\,dP^s(\bar{\rho}^s/\theta^s)$, with P^s the pressure of the sth phase and $\Phi(x,y)$ an external potential. The natural boundary conditions are

$$\mathbf{v}^s\cdot\hat{\mathbf{n}} = 0,\ \mathbf{w}\cdot\hat{\mathbf{n}} = 0,\ \nabla\theta\times\hat{\mathbf{n}} = 0 \ \text{ on } \partial D\,, \qquad [37f]$$

where $\hat{\mathbf{n}}$ is the unit vector lying in the x–y plane in the direction of the outward normal to the boundary ∂D. These boundary conditions imply energy conservation for the quantity H in [42]. Taking the curl of the motion equations [37b] for $\mathbf{v}^s$, rewritten as

$$\partial_t\mathbf{v}^s = \mathbf{v}^s\times\mathrm{curl}\,\mathbf{v}^s - \nabla\left[\frac{1}{2}|\mathbf{v}^s|^2 + h^s\left(\frac{\bar{\rho}^s}{\theta^s}\right) + \Phi(x,y)\right], \qquad [38]$$

and using the continuity equations for $\bar{\rho}^s$ leads to the advected quantities Ω^s, $s = 1,2$, satisfying

$$\left(\frac{\partial}{\partial t} + \mathbf{v}^s\cdot\nabla\right)\Omega^s = 0,\ \Omega^s = (\bar{\rho}^s)^{-1}\hat{\mathbf{z}}\cdot\mathrm{curl}\,\mathbf{v}^s\,, \qquad [39]$$

i.e. Ω^s is constant along the flow lines of the sth species.

In view of [39] for Ω^s and the continuity equation [37a] for $\bar{\rho}^s$, as well as the analogous set up [37c] and [37d] for the pair (θ,σ), the functional

$$C_{F,G}(\Omega^s,\bar{\rho}^s;\,\theta,\sigma) = \int_D dx\,dy\left[\sum_s\bar{\rho}^sF^s(\Omega^s) + \sigma G(\theta)\right] \qquad [40]$$

is conserved by the barotropic planar two-phase multipressure equations for real-valued functions F^s and G such that the integrals exist and the solutions are smooth (Ω^s would be created at a discontinuity either in velocity or in macroscopic density). Another conserved quantity is the following functional, with $\overline{\omega} = \hat{\mathbf{z}} \cdot \text{curl } \mathbf{w}$;

$$C_\psi(\overline{\omega},\theta) = \int_D \mathrm{d}x\,\mathrm{d}y\; \overline{\omega}\psi(\theta) \tag{41}$$

for an arbitrary function $\psi(\theta)$. Conservation of C_ψ is analogous to Kelvin's theorem. Indeed, we find directly that

$$\begin{aligned}\partial_t C_\psi &= -\int_D \mathrm{d}x\,\mathrm{d}y\left[\text{div}(\overline{\omega}\psi(\theta)\mathbf{w}) + \psi(\theta)\,\hat{\mathbf{z}}\cdot\nabla\theta \times \nabla\left(\frac{P^2-P^1}{\sigma}\right)\right] \\ &= -\oint_{\partial D}\left[\overline{\omega}\psi(\theta)\mathbf{w}\cdot\hat{\mathbf{n}} + \psi(\theta)\left(\frac{P^2-P^1}{\sigma}\right)\hat{\mathbf{z}} \times \nabla\theta\cdot\hat{\mathbf{n}}\right]\mathrm{d}s\,,\end{aligned}$$

which vanishes by the boundary conditions [37f]. Kelvin's theorem for the contour on the boundary corresponds to the case $\psi(\theta) = 1$ in [41]. Finally, the total energy of the barotropic planar flow is conserved because it is the Hamiltonian of the system [37], namely,

$$\begin{aligned}H &= \int_D \mathrm{d}x\,\mathrm{d}y\left[\frac{1}{2}\sigma|\mathbf{w}|^2 + \epsilon(\sigma)\right] \\ &+ \sum_{s=1}^{2}\int_D \mathrm{d}x\,\mathrm{d}y\left[\frac{1}{2}\overline{\rho}^s|\mathbf{v}^s|^2 + \overline{\rho}^s e^s\left(\frac{\overline{\rho}^s}{\theta^s}\right) + \overline{\rho}^s\Phi(x,y)\right].\end{aligned} \tag{42}$$

the equilibrium states $\{\overline{\rho}^s_e, \mathbf{v}^s_e, \sigma_e, \theta_e, \mathbf{w}_e\}$ of the system [37] in the x–y plane are the steady, planar, barotropic, two-phase flows. Such steady flows satisfy

$$0 = \text{div}\,\overline{\rho}^s_e\mathbf{v}^s_e\,, \tag{43a}$$

$$0 = -\omega^s_e\hat{\mathbf{z}} \times \mathbf{v}^s_e - \nabla\left[\frac{1}{2}|\mathbf{v}^s_e|^2 + h^s\left(\frac{\overline{\rho}^s_e}{\theta^s_e}\right) + \Phi(x,y)\right], \tag{43b}$$

$$0 = \mathbf{v}^s_e\cdot\nabla\Omega^s_e\,, \tag{43c}$$

$$0 = \text{div}\,\sigma_e\mathbf{w}_e\,, \tag{43d}$$

$$0 = -\overline{\omega}_e\hat{\mathbf{z}} \times \mathbf{w}_e - \nabla\left[\frac{1}{2}|\mathbf{w}_e|^2 + \epsilon'(\sigma_e)\right] + \frac{1}{\sigma_e}(P^2_e - P^1_e)\nabla\theta_e\,, \tag{43e}$$

$$0 = \mathbf{w}_e\cdot\nabla\theta_e\,, \tag{43f}$$

where the quantities

$$\omega^s_e = \hat{\mathbf{z}}\cdot\text{curl}\,\mathbf{v}^s_e = \overline{\rho}^s_e\Omega^s_e\,, \tag{43g}$$

$$\overline{\omega}_e = \hat{\mathbf{z}}\cdot\text{curl}\,\mathbf{w}_e\,, \tag{43h}$$

are the scalar vorticities. The steady equations [43] imply various relations among the equilibrium states. According to [43b] and [43c], the gradient vectors $\nabla[\frac{1}{2}|\mathbf{v}^s_e|^2 + h^s(\overline{\rho}^s_e/\theta^s_e) + \Phi(x,y)]$ and $\nabla\Omega^s_e$ are both orthogonal to the equilibrium species velocity $\mathbf{v}^s_e$ in the plane. Consequently, these two gradient vectors are collinear, provided neither they, nor the velocity vanish. A sufficient condition for this collinearity is the functional relationship for each s:

$$\frac{1}{2}|\mathbf{v}^s_e|^2 + h^s\left(\frac{\overline{\rho}^s_e}{\theta^s_e}\right) + \Phi(x,y) = K^s(\Omega^s_e) \tag{44}$$

for certain functions K^s, $s = 1, 2$. These are called the Bernoulli functions and [44] represents Bernoulli's law for each phase. Likewise, [43e] and [43f] imply collinearity of $\nabla[\frac{1}{2}|\mathbf{w}_e|^2 + \epsilon'(\sigma_e)]$ and $\nabla\theta_e$, for which a sufficient condition is

$$\frac{1}{2}|\mathbf{w}_e|^2 + \epsilon'(\sigma_e) = \overline{K}(\theta_e) \qquad [45]$$

for another Bernoulli function $\overline{K}$

Vector multiplying [43b] by $\hat{\mathbf{z}}$ and using [44] gives

$$\omega_e^s \mathbf{v}_e^s = \hat{\mathbf{z}} \times \nabla K^s(\Omega_e^s)\,, \qquad [46]$$

so that div $\overline{\rho}_e^s\, \mathbf{v}_e^s = 0$ and

$$\frac{K^{s\prime}(\Omega_e^s)}{\Omega_e^s} = \frac{\overline{\rho}_e^s \mathbf{v}_e^s \cdot \hat{\mathbf{z}} \times \nabla \Lambda_e^s}{|\nabla \Omega_e^s|^2}\,, \qquad [47]$$

provided $\omega_e^s \nabla \Omega_e^s \neq 0$. Likewise, vector multiplying [43e] by $\hat{\mathbf{z}}$ and using [45] gives

$$\overline{\omega}_e \mathbf{w}_e = \left[\overline{K}'(\theta_e) + \frac{1}{\sigma_e}(P_e^1 - P_e^2)\right]\hat{\mathbf{z}} \times \nabla\theta_e\,. \qquad [48]$$

For div$(\sigma_e \mathbf{w}_e)$ in [43d] to vanish requires, upon substitution of [48] into [43d], that

$$\hat{\mathbf{z}} \times \nabla\theta_e \cdot \nabla\{(\overline{\omega}_e)^{-1}[\sigma_e \overline{K}'(\theta_e) + P_e^1 - P_e^2]\} = 0 \qquad [49]$$

For this, it is sufficient that

$$\frac{1}{\overline{\omega}_e}[\sigma_e K'(\theta_e) + P_e^1 - P_e^2] = L(\theta_e) \qquad [50]$$

for some function $L(\theta_e)$. Relation [50] is analogous to Long's equation in stratified fluid flow (see e.g. Abarbanel *et al*. 1984; 1986). Note that [48] and [50] imply

$$\sigma_e \mathbf{w}_e = L(\theta_e)\hat{\mathbf{z}} \times \nabla\theta_e\,, \qquad [51]$$

so that

$$L(\theta_e) = \frac{\sigma_e \mathbf{w}_e \cdot \hat{\mathbf{z}} \times \nabla\theta_e}{|\nabla\theta_e|^2}\,. \qquad [52]$$

These relations among equilibrium states will be useful in proving the following proposition, analogous to that in the previous section for the three-dimensional case.

Proposition. Equilibrium solutions $\{\overline{\rho}_e^s, \mathbf{v}_e^s, \sigma_e, \mathbf{w}_e, \theta_e\}$ of the planar barotropic two-phase equations [37a–e] satisfying boundary conditions [37f] and $\omega_e^s \overline{\omega}_e \neq 0$, $\nabla\theta_e \neq 0$ are critical points of the functional

$$H_C = H + C_{F,G} + C_\psi + \Lambda \int \mathrm{d}x\mathrm{d}y\, \overline{\omega} + \sum_s \lambda^s \int \mathrm{d}x\mathrm{d}y\, \omega^s$$

composed of the sum of the conserved quantities given in [40]–[42], provided F^s, G, and ψ are determined by

$$K^s(\Omega_e^s) + F^s(\Omega_e^s) - \Omega_e^s F^{s\prime}(\Omega_e^s) = 0\,, \qquad [53a]$$

$$G(\theta_e) = -\overline{K}(\theta_e)\,, \qquad [53b]$$

$$\psi'(\theta_e) = L(\theta_e)\,, \qquad [53c]$$

where, for particular steady states, K^s and $\overline{K}$ are the Bernoulli functions in [44] and [45], respectively, L is the Long function in [50], and where Λ and λ^s are constants determined by the constant values of θ_e and Ω_e^s on the boundary ∂D. Conversely, a critical point of H_C is an equilibrium solution.

Proof. The conserved functional H_C in the proposition is given explicitly by

$$H_C = \int_D \mathrm{d}x\mathrm{d}y\left[\frac{1}{2}\sigma|\mathbf{w}|^2 + \epsilon(\sigma) + \sigma G(\theta) + \overline{\omega}(\psi(\theta) + \Lambda)\right]$$
$$+ \sum_{s=1}^{2}\int_D \mathrm{d}x\mathrm{d}y\left[\frac{1}{2}\overline{\rho}^s|\mathbf{v}^s|^2 + \overline{\rho}^s e^s\left(\frac{\overline{\rho}^s}{\theta^s}\right) + \overline{\rho}^s\Phi(\mathbf{x}) + \overline{\rho}^s F^s(\Omega^2) + \lambda^s\hat{\mathbf{z}}\cdot\mathrm{curl}\,\mathbf{v}^s\right],$$

where D is the domain of flow and λ^s, Λ are constants separated out from the functions F^s and ψ for convenience. After integrating by parts, the first variation of H_C evaluated at the equilibrium state becomes

$$\begin{aligned}\delta H_C|_e := {} & DH_C(\overline{\rho}_e^s, \mathbf{v}_e^s, \sigma_e, \mathbf{w}_e, \theta_e)\cdot(\delta\overline{\rho}^s, \delta\mathbf{v}^s, \delta\sigma, \delta\mathbf{w}, \delta\theta)\\
= {} & \int_D \mathrm{d}x\mathrm{d}y\left\{\delta\sigma\left[\frac{1}{2}|\mathbf{w}_e^2| + \epsilon'(\sigma_e) + G(\theta_e)\right]\right.\\
& + \delta\mathbf{w}\cdot[\sigma_e\mathbf{w}_e + \psi'(\theta_e)\nabla\theta_e\times\hat{\mathbf{z}}]\\
& + \sum_s \delta\mathbf{v}^s\cdot[\overline{\rho}_e^s\mathbf{v}_e^s + F^{s\prime\prime}(\Omega_e^s)\nabla\Omega_e^s\times\hat{\mathbf{z}}]\\
& + \sum_s \delta\overline{\rho}^s\left[\frac{1}{2}|\mathbf{v}_e^s|^2 + h^s\left(\frac{\overline{\rho}_e^s}{\theta_e^s}\right) + \Phi(\mathbf{x}) + F^s(\Omega_e^s) - \Omega_e^s F^{s\prime}(\Omega_e^s)\right]\\
& \left. + \delta\theta\left[P_e^2 - P_e^1 + \sigma_e G'(\theta_e) + \overline{\omega}_e\psi'(\theta_e)\right]\right\}\\
& - \oint_{\partial D}(\Lambda + \psi(\theta_e))\hat{\mathbf{z}}\times\delta\mathbf{w}\cdot\hat{\mathbf{n}}\,\mathrm{d}s\\
& - \sum_s\oint_{\partial D}[\lambda^s + F^{s\prime}(\Omega_e^s)]\hat{\mathbf{z}}\times\delta\mathbf{v}\cdot\hat{\mathbf{n}},\end{aligned} \qquad [54]$$

where $\mathrm{d}s$ is the line element on the boundary ∂D and $\hat{\mathbf{n}}$ is its unit normal vector in the plane. The boundary terms each vanish upon choosing Λ and λ^s so that

$$\Lambda + \psi(\theta_e)|_{\partial D} = 0,$$

$$\lambda^s + F^{s\prime}(\Omega_e^s)|_{\partial D} = 0,$$

which is possible since θ_e and Ω_e^s are constants on ∂D by [43c], [43f], and the boundary conditions [37f]. The remaining coefficients in [54] each vanish upon imposing the conditions of the proposition and the equilibrium relations: the $\delta\overline{\rho}^s$ coefficient vanishes by [53a] and [44]; that of $\delta\sigma$ vanishes by [53b] and [45]; that of $\delta\mathbf{v}^s$ by [53a] and [46]; that of $\delta\mathbf{w}$ by [53b] and [51]; and, finally, the $\delta\theta$ coefficient vanishes by [53c] and [50]. Conversely, if δH_C vanishes for some flow and the relations [53] hold, one derives the relations [37a–f], so that the flow is stationary.

The second variation of H_C at the equilibrium state is

$$\delta^2 H_C = D^2 H_C(\bar{\rho}_e^s, \mathbf{v}_e^s, \sigma_e, \mathbf{w}_e, \theta_e)\cdot(\delta\bar{\rho}^s, \delta\mathbf{v}^s, \delta\sigma, \delta\mathbf{w}, \delta\theta)^2$$

$$= \int_D \mathrm{d}x\mathrm{d}y\{\sigma_e|\delta\mathbf{w}|^2 + 2\mathbf{w}_e\cdot\delta\mathbf{w}\delta\sigma$$

$$+ \epsilon''(\sigma_e)(\delta\sigma)^2 + 2G'(\theta_e)\delta\theta\delta\sigma$$

$$+ [\sigma_e G''(\theta_e) + \bar{\omega}_e\psi''(\theta_e)](\delta\theta)^2 + 2\psi'(\theta_e)\delta\omega\delta\theta\} \qquad [55]$$

$$+ \sum_{s=1}^{2}\int_D \mathrm{d}x\mathrm{d}y\left[\bar{\rho}_e^s|\delta\mathbf{v}^s|^2 + 2\mathbf{v}_e^s\cdot\delta\mathbf{v}^s\delta\bar{\rho}^s\right.$$

$$\left. + (\bar{\rho}_e^s)^{-1}(\theta_e^s)^2(C_e^s)^2\left(\delta\left(\frac{\bar{\rho}^s}{\theta^s}\right)\right)^2 + \bar{\rho}_e^s F^{s\prime\prime}(\Omega_e^s)(\delta\Omega^s)^2\right],$$

where $\delta\bar{\omega} = \mathbf{z}\cdot\mathrm{curl}\,\delta\mathbf{w}$ and $\delta^2\Omega^s = -2(\rho_e^s)^{-1}\delta\Omega^s\delta\rho^s$ have been used, and $(c_e^s)^2 = \partial P_e^s/\partial(\bar{\rho}_e^s/\theta_e^s)$. If $\delta^2 H_C$ in [55] has a definite sign, it must be positive, since $\bar{\rho}_e^s > 0$.

Equilibrium states for which $\delta^2 H_C$ is positive definite are linearly Lyapunov stable. That is, a positive definite $\delta^2 H_C$ can be used as a norm that bounds the deviation from equilibrium under the linearized dynamics. Indeed, the value of $\delta^2 H_C$ is preserved by the dynamics of the equations linearized about the equilibrium state, since $\delta^2 H_C$ is the Hamiltonian for the linearized dynamics (see e.g. Abarbanel *et al.* 1986, appendix C). Thus, for positive definite $\delta^2 H_C$, a perturbed state initially near the equilibrium state will remain near to it in the sense of the norm defined by $\delta^2 H_C$. This means the equilibrium is Lyapunov stable under the linearized dynamics whenever $\delta^2 H_C$ is positive definite. (A conserved norm can be defined for negative definite $\delta^2 H_C$ as well, of course, but this is not the situation in the present case.)

If, further, the conserved functional H_C is convex, then the equilibrium state will be nonlinearly Lyapunov stable, i.e. finite amplitude deviations from equilibrium will remain near the equilibrium in the sense of a norm that bounds the deviations of H_C from its equilibrium value. See Holm *et al.* (1985) for further theoretical details and examples of how to establish nonlinear Lyapunov stability conditions using convexity arguments and the Hamiltonian formalism of ideal fluids and plasmas.

Sufficient conditions for $\delta^2 H_C$ in [55] to be positive definite are found by determining when the integrand in [55] is positive definite as an algebraic quadratic form in ($\delta\sigma$, $\delta\mathbf{w}$, $\delta\theta$, $\delta\bar{\rho}^s$, $\delta\mathbf{v}^s$). Expanding $(\delta(\bar{\rho}^s/\theta^s))^2$, using $\Sigma_s\delta\theta^s = 0$, and completing the square among the $\delta\Omega^s$ terms in [55] gives

$$\delta^2 H_C = \int_D \mathrm{d}x\mathrm{d}y\left\{\left[\sum_{s=1}^{2}\bar{\rho}_e^s F^{s\prime\prime}(\Omega_e^s)(\delta\Omega^s)^2\right] + \sigma_e|\delta\mathbf{w}|^2 + T\right\}. \qquad [56]$$

Here T is defined to be

$$T = \int_D \mathrm{d}x\mathrm{d}y\left\{\epsilon''(\sigma_e)(\delta\sigma)^2 + 2G''(\theta_e)\delta\sigma\delta\theta + A(\delta\theta)^2 + 2\mathbf{w}_e\cdot\delta\mathbf{w}\delta\sigma\right.$$

$$+ 2\psi'_e(\theta)\delta\bar{\omega}\delta\theta + 2\left[\sum_{s=1}^{2}(-1)^s\alpha^s\delta\rho^s\right]\delta\theta \qquad [57]$$

$$\left. + \sum_{s=1}^{2}\left[\gamma^s(\delta\bar{\rho}^s)^2 + 2\mathbf{v}_e^s\cdot\delta\mathbf{v}^s\delta\bar{\rho}^s + \bar{\rho}_e^s|\delta\mathbf{v}^s|^2\right]\right\},$$

where the following coefficients have been introduced:

$$
\begin{aligned}
A &= \bar{\omega}_e \psi''(\theta_e) + \sigma_e G''(\theta_e) + \sum_{s=1}^{2} \frac{\bar{\rho}_e^s (c_e^s)^2}{(\theta_e^s)^2}, \\
\alpha^s &= \frac{(c_e^s)^2}{\theta_e^s}, \\
\gamma^s &= \frac{(c_e^s)^2}{\bar{\rho}_e^s}.
\end{aligned} \qquad [58]
$$

Let $F^{s\prime\prime}(\Omega_e^s) > 0$ and $\sigma_e > 0$ in [57]. Then for $\delta^2 H_C$ in [56] to be positive definite, the quadratic form T in the integrand [57] needs to be positive semidefinite. This is possible, however, only if $\psi'(\theta_e) = 0$, since the only term in T containing $\delta\bar{\omega}$ is $2\psi'(\theta_e)\,\delta\bar{\omega}\,\delta\theta$. By [53c], [43f], and [52] it follows that $\mathbf{w}_e = 0$. Upon setting $\psi'(\theta_e) = 0$ and $\mathbf{w}_e = 0$ in [56] we find, by sequentially completing squares in [57], that $\delta^2 H_C$ is positive definite under the conditions

$$F^{s\prime\prime}(\Omega_e^s) > 0, \qquad [59]$$

$$\bar{\rho}_e^s > 0, \qquad [60a]$$

$$\sigma_e > 0, \qquad [60b]$$

$$\epsilon''(\sigma_e) > 0, \qquad [60c]$$

$$(c_e^s)^2 - |\mathbf{v}_e^s|^2 > 0, \qquad [61]$$

$$\epsilon''(\sigma_e)\left[A - \sum_{s=1}^{2} \frac{\bar{\rho}_e^s (\alpha^s)^2}{(c_e^s)^2 - |v_e^s|^2}\right] - (G'(\theta_e))^2 > 0, \qquad [62]$$

where A, α^s, and γ^s are given in [58]. Stability conditions [60] are self-explanatory. Condition [61] requires that the flow be subsonic for each species. The last condition [62] can be rewritten using [58], [53b], and [48] for $\mathbf{w}_e = 0$ as

$$\epsilon''(\sigma_e)\left[\sigma_e \frac{d}{d\theta}\left(\frac{P_e^1 - P_e^2}{\sigma}\right) - \sum_{s=1}^{2} \frac{\bar{\rho}_e^s |\mathbf{v}_e^s|^2/(\theta_e^s)^2}{1 - |\mathbf{v}_e^s|^2/(c_e^s)^2}\right] - \left(\frac{P_e^1 - P_e^2}{\sigma_e}\right) > 0. \qquad [62']$$

Condition [59] can be understood by noting that [47] and [53a] imply

$$F^{s\prime\prime}(\Omega_e^s) = \frac{K^{s\prime}(\Omega_e^s)}{\Omega_e^s} = \frac{\bar{\rho}_e^s \mathbf{v}_e^s \cdot \hat{\mathbf{z}} \times \nabla \Omega_e^s}{|\nabla \Omega_e^s|^2}. \qquad [63]$$

Consequently, condition [59] holds when $\mathbf{v}_e^s$, $\hat{\mathbf{z}}$, and $\nabla\Omega_e^s$ form a right-handed triad.

Example. Subsonic shear flows. A steady equilibrium solution of the planar barotropic two-phase equations [43] in the strip $\{(x,y) \in \mathbb{R}^2 | Y_1 \le y \le Y^2\}$ is a plane-parallel flow along the x axis admitting the velocity profiles $\mathbf{v}_e^s = (v^s(y), 0)$, densities $\bar{\rho}_e^s = \bar{\rho}^s(y)$, volume fraction $\theta_e = \theta(y)$, $\sigma_e = \sigma(y)$, and $\mathbf{w}_e = 0$. Given the profiles $v^s(y)$ and $\bar{\rho}^s(y)$ one finds that

$$
\begin{aligned}
\Omega^s(y) &= (\bar{\rho}^s)^{-1}\hat{\mathbf{z}} \cdot \operatorname{curl} v^s(y)\hat{\mathbf{x}} \\
&= -(\bar{\rho}^s)^{-1} v^{s\prime}(y).
\end{aligned} \qquad [64]
$$

For the present case, $-\nabla\Omega_e^s = (v^{s\prime}/\bar{\rho}^s)'\,\hat{\mathbf{y}}$, so [59] requires that

$$F^{s\prime\prime}(\Omega_e^s) = \frac{\bar{\rho}^s v^s}{(v^{s\prime}/\bar{\rho}^s)'} > 0. \qquad [65]$$

When $\bar{\rho}^s$ = const for $s = 1, 2$, the stability condition [65] reduces to the requirement $v^s v^{s\prime\prime} > 0$ for both species, i.e. it reverts to the classical inflection point criterion due to Rayleigh. From this example, the conclusion is that for stability of a multiphase flow, both of the species velocities must satisfy [65], i.e. both species must be individually stable according to classical fluid mechanics, and, in addition, the combined properties of these individually stable flows must be governed by [62′]

7 CONCLUSION

We have regularized single-pressure multicomponent fluid dynamics by extending its Poisson bracket and Hamiltonian to include multiple pressures in such a way that the known Hamiltonian structure of the multispecies fluid equations is recovered in the absence of interface variables. The motion equations for the extended model include transport equations for interface variables that can be interpreted as belonging to an additional fluid with zero volume fraction, but finite mass density corresponding to the interfacial inertia that enables the fluid pressures to differ across the interface. We have shown that the extended model provides a hyperbolic regularization of the ill-posed single-pressure model. Moreover, we have shown that Lyapunov stable equilibria exist for the extended system in three dimensions, and found explicit Lyapunov stability conditions for the extended model in two dimensions by characterizing a certain class of equilibria for the model as critical points of conserved Lyapunov functionals.

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